

# The Heavy Quark Potential as a Function of Shear Viscosity at Strong Coupling

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## Abstract

We determine finite temperature corrections to the heavy-quark (static) potential as a function of the shear viscosity to entropy density ratio in a strongly coupled, large- $N_c$  conformal field theory dual to five-dimensional Gauss-Bonnet gravity. We find that these corrections are even smaller than those predicted by perturbative QCD at distances relevant for small bound states in a deconfined plasma. Obtaining the dominant temperature and viscosity dependence of quarkonium binding energies will require a theory where conformal invariance is broken in such a way that the free energy associated with a single heavy quark is not just a pure entropy contribution.

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## I. INTRODUCTION

The Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence [1] relates correlation functions of local conformal fields in 4-dimensional strongly-coupled non-Abelian plasmas to the asymptotic behavior of fields defined in weakly-coupled, low energy effective string theories in higher dimensions. The conformal theories involved in the AdS/CFT correspondence depend on the number of colors  $N_c$  and on the t'Hooft coupling  $\lambda = g^2 N_c$ . In particular, when  $N_c \rightarrow \infty$  and  $g \rightarrow 0$  but  $\lambda \gg 1$ , the strongly coupled CFT in  $D = 4$  is dual to a weakly-coupled  $D = 10$  theory of (super)gravity. The equivalence of strongly-coupled 4-dimensional  $\mathcal{N} = 4$  Supersymmetric Yang-Mills (SYM) to type IIB string theory on  $\text{AdS}_5 \otimes S^5$  [1] has led to new insight into the non-perturbative dynamics of strongly-coupled gauge theories at finite temperature [2]. For instance, it was shown that the shear viscosity to entropy density ratio satisfies  $4\pi \eta/s \geq 1$  in all gauge theories dual to supergravity [3].

In general, due to the colossal number of possible vacua in the current version of the string landscape [4], one may expect that higher derivative corrections to the gravity sector in  $\text{AdS}_5$  can occur. Using the relation  $\sqrt{\lambda} = R^2/\alpha'$  (where  $R$  is the radius of  $\text{AdS}_5$ ), the  $\mathcal{O}(\alpha')$  expansion in type IIB string theory becomes an expansion in powers of  $1/\sqrt{\lambda}$  in the SYM theory. Quartic corrections are known to be present in closed superstring theory [5] (supersymmetry excludes terms corresponding to cubic powers of Riemann tensors [6]). In fact, it was shown in [7] that the leading corrections to the type IIB tree level effective action are due to terms of the form  $\alpha'^3 \mathcal{R}^4$ , which in turn generate positive corrections of  $\mathcal{O}(\lambda^{-3/2})$  to  $\eta/s$  that preserve the viscosity bound [8].

On the other hand, curvature squared interactions can be induced in the effective 5-dimensional gravity sector by including the world-volume action of D7-branes [9, 10], which are normally used in the holographic description of the fundamental flavors in the dual gauge theory [11]. It was shown in Refs. [12, 13, 14] that 5-dimensional gravity theories with curvature squared terms in the action are dual to 4-dimensional superconformal theories where  $\eta/s$  can be lower than  $1/(4\pi)$ . Additionally,  $\eta/s$  was found to be a very simple analytical function of the new parameter associated with the high derivative contributions (which is fully determined by the central charges of the CFT). In fact, the very detailed study done in Ref. [10] confirmed (and extended) the initial claim made in Ref. [14] that the

viscosity bound should be violated in superconformal theories with different central charges.

In this paper, we use the gravity dual discussed in [12, 13], which includes  $\mathcal{R}^2$  corrections, to calculate the dependence of the heavy quark potential at (moderately) short distances on  $\eta/s$  in a strongly-coupled non-Abelian plasma. The  $Q\bar{Q}$  potential at finite temperature can be calculated as a power series in  $LT \ll 1$ , where  $L$  is the spatial distance between the heavy quarks. It is shown that the potential energy increases with  $\eta/s$  and that the effective medium-induced “screening” of the attractive potential decreases much more rapidly with increasing viscosity and quark mass at strong rather than at weak t’Hooft coupling.

We would like to point out that we have limited our discussion to the class of gravity theories that are dual to strongly coupled superconformal gauge theories with non-equal central charges such as those in [12, 13, 14]. Other corrections originating from  $\mathcal{R}^4$  terms are not included in our discussion. The combined effects of  $\mathcal{R}^2$  and  $\mathcal{R}^4$  corrections to the heavy quark potential are left for future work.

This paper is organized as follows. In the next section we review how curvature squared corrections to the effective 5-dimensional gravitational action affect the black brane horizon and, consequently, lead to a modification of both thermodynamic and transport properties of the dual D=4 CFT. In Section III we show how these corrections affect the heavy quark potential at zero and at finite temperature. Once the heavy quark potential is known, in Section IV we determine the resulting binding energy of the  $Q\bar{Q}$  ground state and its dependence on  $\eta/s$ . We close with a summary and outlook.

## II. $R^2$ CORRECTIONS TO THE 5-DIMENSIONAL GRAVITATIONAL ACTION

The effects of curvature squared corrections can be described by the general action [12, 14]

$$S = \frac{1}{16\pi G_5} \int d^5x \sqrt{-G} \left[ \mathcal{R} + \frac{12}{R^2} + R^2 (c_1 \mathcal{R}^2 + c_2 \mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu} + c_3 \mathcal{R}_{\mu\nu\lambda\rho} \mathcal{R}^{\mu\nu\lambda\rho}) \right] \quad (1)$$

where  $G_5 = \pi R^3/(2N_c^2)$  and  $R$  is the radius of  $\text{AdS}_5$  at leading order in  $c_i$ . The coefficients  $c_i$  are expected to be of  $\mathcal{O}(\alpha')$ , which means that  $\lim_{\lambda \rightarrow \infty} c_i = 0$ . However, at this order only  $c_3$  is unambiguous because the coefficients  $c_1$  and  $c_2$  can be arbitrarily modified via a simple redefinition of the metric [10, 12, 14].

The shear viscosity-to-entropy ratio, to first-order in  $c_i$ , was found to be [12, 14]

$$\frac{\eta}{s} = \frac{1}{4\pi} (1 - 8c_3) + \mathcal{O}(c_i^2). \quad (2)$$

Therefore, the viscosity bound is violated when  $c_3 > 0$ . For 4-dimensional CFTs with  $\text{AdS}_5$  gravity duals in the limit where  $\lambda \gg 1$  and  $N_c \rightarrow \infty$ , one has  $c_3 = (c - a)/(8c) + \mathcal{O}(1/N_c^2)$ , where  $a$  and  $c$  are the central charges of the CFT [15]. For  $\mathcal{N} = 4$   $SU(N_c)$  SYM  $c = a$  exactly and the bound is preserved, although there are superconformal theories in which  $\eta/s$  receives a correction of  $\mathcal{O}(1/N_c)$  that violates the bound [10, 14].

Gauss-Bonnet (GB) gravity [16] is a special case of the general action in (1) where  $c_2 = -4c_1$  and  $c_1 = c_3 = \lambda_{GB}/2$ , which gives the action

$$S_{GB} = \frac{1}{16\pi G_5} \int d^5x \sqrt{-G} \left[ \mathcal{R} + \frac{12}{R^2} + \frac{\lambda_{GB}}{2} R^2 (\mathcal{R}^2 - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}_{\mu\nu\lambda\rho}\mathcal{R}^{\mu\nu\lambda\rho}) \right]. \quad (3)$$

For this particular combination of coefficients the metric fluctuations around a given background have the same quadratic kinetic terms as Einstein gravity (higher derivative terms cancel [16]). Another interesting feature of GB gravity is that an exact black brane solution [17] is known for  $\lambda_{GB} \in (-\infty, 1/4)$

$$ds^2 = -a^2 f_{GB}(U) dt^2 + \frac{U^2}{R^2} d\vec{x}^2 + \frac{dU^2}{f_{GB}(U)}, \quad (4)$$

where  $a^2 = \frac{1}{2} (1 + \sqrt{1 - 4\lambda_{GB}})$  and

$$f_{GB}(U) = \frac{U^2}{R^2} \frac{1}{2\lambda_{GB}} \left[ 1 - \sqrt{1 - 4\lambda_{GB} \left( 1 - \frac{U_h^4}{U^4} \right)} \right]. \quad (5)$$

The parameter  $a$  has the form above to make sure that the speed of light at the boundary ( $U \rightarrow \infty$ ) is unity. The horizon of the GB black brane is the simple root of  $f_{GB}$  located at  $U = U_h$ . The plasma temperature in this case is

$$T = a \frac{U_h}{\pi R^2} \quad (6)$$

whereas the entropy density is

$$s = \frac{1}{4G_5} \left( \frac{U_h}{R} \right)^3 = \frac{N_c^2 \pi^2 T^3}{2a^3}. \quad (7)$$

At this point the only formal constraint on the Gauss-Bonnet coupling is that  $\lambda_{GB} \in (-\infty, 1/4)$ . However, it was shown in [12] that

$$\frac{\eta}{s} = \frac{1}{4\pi} (1 - 4\lambda_{GB}), \quad (8)$$

to all orders in  $\lambda_{GB}$ . However,  $\lambda_{GB} \leq 9/100$  or, equivalently,  $4\pi \eta/s \geq 16/25$  in order to avoid causality violation in the boundary [13]. In any case, as was mentioned above,

one should expect that  $|\lambda_{GB}| \sim \alpha'/R^2 \ll 1$  at strong  $t'$  Hooft coupling. In this paper we take  $\lambda_{GB}$  to be a free parameter which parameterizes the ratio of shear viscosity to entropy density.

Note that the AdS radius in the GB geometry is not just  $R$  but  $aR$  [12]. Thus, here we assume that the effective  $t'$  Hooft coupling of the 4d CFT dual to the GB theory in Eq. (3) is  $\lambda = R^4 a^4 / \alpha'^2$ . Moreover, the  $t'$  Hooft coupling is assumed to be large such that qualitatively meaningful results can be obtained at leading order in  $\lambda$ , but finite [18]. The heavy-quark potential in the strongly-coupled CFT only permits non-relativistic bound states, and indeed bound states where the quarks are not localized over distances smaller than their Compton wavelength, if the  $t'$  Hooft coupling is not too large; c.f. Section IV.

### III. $R^2$ CORRECTIONS TO THE HEAVY QUARK POTENTIAL

We will be interested in the Wilson loop operator<sup>1</sup>

$$W(C) = \frac{1}{N_c} \text{Tr} P e^{i \int A_\mu dx^\mu} \quad (9)$$

where  $C$  denotes a closed loop in spacetime and the trace is over the fundamental representation of  $SU(N_c)$ . We consider a rectangular loop with one direction along the time coordinate  $t$  and spatial extension  $L$ . In the asymptotic limit  $t \rightarrow \infty$ , the vacuum expectation value of the loop defines a static potential via

$$\langle W(C) \rangle \sim e^{-t V_{Q\bar{Q}}(L)} . \quad (10)$$

Using somewhat loose language we call this the “heavy-quark potential”.

The expectation value of  $W(C)$  can be calculated in the strongly coupled  $\mathcal{N} = 4$  SYM theory using supergravity [19, 20]. According to the AdS/CFT correspondence, an infinitely massive heavy quark in the fundamental representation of  $SU(N_c)$  in the  $\mathcal{N} = 4$  SYM theory is dual to a classical string in the bulk that hangs down from a probe brane at the boundary of  $\text{AdS}_5$  [19, 20] when  $N_c \rightarrow \infty$  and  $\lambda \gg 1$  (supergravity approximation). Within this approximation, the dynamics of the string (in Euclidean space) is given by the classical

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<sup>1</sup> Even though the string dynamics can be in principle fully 10-dimensional, here we consider only the dynamics corresponding to the 5 non-compact coordinates.

Nambu-Goto action

$$S_{NG} = \frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{\det h_{ab}} \quad (11)$$

where  $h_{ab} = G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu$  ( $a, b = 1, 2$ ),  $G_{\mu\nu}$  is the background bulk metric,  $\sigma^a = (\tau, \sigma)$  are the internal world sheet coordinates, and  $X^\mu = X^\mu(\tau, \sigma)$  is the embedding of the string in the 10-dimensional spacetime. In the supergravity approximation, since the endpoint of the string at the boundary carries fundamental charge, it is natural to assume that

$$\langle W(C) \rangle \sim e^{-\Delta S_{NG}}, \quad (12)$$

where the loop  $C$  is defined at the boundary of  $\text{AdS}_5$ . In the equation above  $\Delta S_{NG}$  is the regularized action, which comes about after subtracting the infinite self-energy associated with two independent and infinitely massive quarks (two straight lines that extend from  $U = 0$  to  $U \rightarrow \infty$ ). Note that this is consistent with the ideas behind holographic renormalization [21]. The configuration that minimizes the action is a curve that connects the string endpoints at the boundary and has a minimum at some  $U_*$  in  $\text{AdS}_5$  [19, 20].

The potential for  $\mathcal{N} = 4$  SYM has the following simple analytical form [19]

$$V_{Q\bar{Q}}(L) = \frac{\Delta S_{NG}}{t} = -\frac{4\pi^2 \sqrt{\lambda}}{\Gamma(1/4)^4} \frac{1}{L}. \quad (13)$$

The  $\sim 1/L$  dependence is due to the conformal invariance of the theory. Also, the potential is non-analytic in  $\lambda$  while the standard short-distance potential in perturbative QCD (pQCD) is, of course, to leading order proportional to the coupling

$$V_{Q\bar{Q}}(L) \Big|_{QCD} = -\frac{g_{QCD}^2 C_F}{4\pi L} \simeq -\frac{\lambda_{QCD}}{8\pi L}, \quad (14)$$

where the latter form applies at large  $N_c$  and  $\lambda_{QCD} = g_{QCD}^2 N_c$ .

One can generalize the calculations performed in Refs. [19, 20] to include the effects from curvature squared corrections given by, for instance, the Gauss-Bonnet theory in Eq. (3). The equations at zero and at finite temperature are very similar and, thus, here we will derive the general form of the equations and only later work out the necessary details for each case.

In general, we have

$$\det h_{ab} = X'^2 \cdot \dot{X}^2 - (\dot{X} \cdot X')^2 \quad (15)$$

where  $X'^\mu(\tau, \sigma) = \partial_\sigma X^\mu(\tau, \sigma)$  and  $\dot{X}^\mu(\tau, \sigma) = \partial_\tau X^\mu(\tau, \sigma)$ . We choose a gauge where the coordinates of our static string are  $X^\mu = (t, x, 0, 0, U(x))$ , where  $\tau = t$  and  $\sigma = x$ . Note

that we use the Euclidean version of eq. (4) and, thus, at finite temperature the fields are periodic in time with a period equal to  $1/T$ . In this case, one finds

$$S_{NG} = a \frac{t}{2\pi\alpha'} \int dx \sqrt{f_{GB}(U(x)) \frac{U^2(x)}{R^2} + U'^2(x)} . \quad (16)$$

Note the presence of the prefactor  $a(\lambda_{GB})$  in the equation above. The Hamiltonian density associated with this action is

$$H_{NG}(x) = -\frac{U^2}{R^2} \frac{f_{GB}(U)}{\sqrt{f_{GB}(U) \frac{U^2}{R^2} + U'^2}} , \quad (17)$$

which is invariant under translations in  $x$ . In what follows we denote the minimum of the U-shaped string at  $x_* = 0$  as  $U_*$ . One can then compute  $H_{NG}(x_*)$

$$H_{NG}(x_*) = -\sqrt{f_{GB}(U_*) \frac{U_*^2}{R^2}} , \quad (18)$$

which due to the translational symmetry is equal to  $H_{NG}(x)$  at any  $x$ . This allows us to solve for  $x = x(U)$ :

$$\begin{aligned} x(U) = & \frac{R^2}{U_*} \left[ 2\lambda_{GB} \left( 1 - \sqrt{1 - 4\lambda_{GB} \varepsilon} \right) \right]^{1/2} \int_1^{U/U_*} dy \left\{ \left[ y^4 - y^2 \sqrt{y^4 - 4\lambda_{GB} (y^4 - 1 + \varepsilon)} \right]^2 \right. \\ & \left. - \left[ y^4 - y^2 \sqrt{y^4 - 4\lambda_{GB} (y^4 - 1 + \varepsilon)} \right] \left[ 1 - \sqrt{1 - 4\lambda_{GB} \varepsilon} \right] \right\}^{-1/2} , \end{aligned} \quad (19)$$

where  $y_* \equiv U_h/U_*$  and  $\varepsilon \equiv 1 - y_*^4$ . One of the string endpoints is located at  $x = -L/2$  while the other one is at  $x = L/2$ . Thus,  $U_*$  is related to  $L$  via

$$\begin{aligned} \frac{L}{2} = & \frac{R^2}{U_*} \left[ 2\lambda_{GB} \left( 1 - \sqrt{1 - 4\lambda_{GB} \varepsilon} \right) \right]^{1/2} \int_1^\infty dy \left\{ \left[ y^4 - y^2 \sqrt{y^4 - 4\lambda_{GB} (y^4 - 1 + \varepsilon)} \right]^2 \right. \\ & \left. - \left[ y^4 - y^2 \sqrt{y^4 - 4\lambda_{GB} (y^4 - 1 + \varepsilon)} \right] \left[ 1 - \sqrt{1 - 4\lambda_{GB} \varepsilon} \right] \right\}^{-1/2} . \end{aligned} \quad (20)$$

Moreover, one can show that the regularized action is given by

$$\frac{1}{2} \Delta S_{NG} = \frac{t a}{2\pi \alpha'} \int_{U_*}^\infty dU \left[ 1 + \frac{1}{\frac{f_{GB}(U)U^2}{f_{GB}(U_*)U_*^2} - 1} \right]^{1/2} - \frac{t a}{2\pi \alpha'} \int^\infty dU \quad (21)$$

$$= \frac{t a}{2\pi \alpha'} U_* \int_1^\infty dy \left\{ \left[ 1 + \frac{1}{\frac{f_{GB}(y)y^2}{f_{GB}(1)} - 1} \right]^{1/2} - 1 \right\} - \frac{t a}{2\pi \alpha'} U_* , \quad (22)$$

where  $y = U/U_*$  and

$$f_{GB}(y) = \frac{y^2}{R^2} \frac{U_h^2}{2\lambda_{GB}} \left[ 1 - \sqrt{1 - 4\lambda_{GB} \left( 1 - \frac{y_h^4}{y^4} \right)} \right] . \quad (23)$$

We have regularized the action (21) by subtracting the contribution of a straight string hanging down from the boundary (corresponding to the infinite mass of the source). This also subtracts a finite part of the action as determined by the lower limit of the second integral from Eq. (21). We choose to subtract (twice) the action at  $T = 0$ , corresponding to a straight string from  $U = \infty$  to  $U = 0$ . The free energy of the  $Q\bar{Q}$  pair is therefore identified with the entire temperature-dependent contribution to the action.

At finite temperature, the free energy due to the heavy quarks (in a color-singlet state) is given by the three-dimensional action of the Wilson loop,

$$F_{Q\bar{Q}} = T\Delta S_{NG} . \quad (24)$$

$F_{Q\bar{Q}}$  should not be interpreted as the heavy-quark potential at finite temperature because it also contains an entropy contribution [22, 23] (especially at large separation  $L \rightarrow \infty$  where  $F_\infty$  coincides with twice the free energy due to a single heavy quark in the plasma; see discussion below). We remove this entropy contribution at all  $L$  by defining

$$V_{Q\bar{Q}} = F_{Q\bar{Q}} - T \frac{\partial F_{Q\bar{Q}}}{\partial T} . \quad (25)$$

Thus,  $V_{Q\bar{Q}}$  coincides with  $F_{Q\bar{Q}}$  at short distances (where temperature effects are absent) but approaches the internal energy as  $L \rightarrow \infty$  [23].

### A. Heavy Quark Potential in the Vacuum

The potential in the vacuum can be calculated to all orders in  $\lambda_{GB}$ . In fact, when  $T \rightarrow 0$  Eq. (20) can be easily solved for  $U_*$

$$U_* = a(\lambda_{GB}) \frac{2R^2}{L} \frac{\sqrt{2}\pi^{3/2}}{\Gamma(1/4)^2} . \quad (26)$$

This can be expressed as

$$U_* = a(\lambda_{GB}) U_* \Big|_{\text{Maldacena}} \quad (27)$$

where  $U_* \Big|_{\text{Maldacena}}$  is the result found in [19, 20]. Thus, we see that the bottom of the U-shaped string approaches the boundary when  $\lambda_{GB}$  goes from  $1/4$  to  $-\infty$ . The action for this configuration is

$$\Delta S_{NG} = -\frac{t}{L} \frac{4\pi^2\sqrt{\lambda}}{\Gamma(1/4)^4} \quad (28)$$



where we used the previous definition  $\sqrt{\lambda} = R^2 a^2 / \alpha'$ . Thus, the potential energy is

$$V_{Q\bar{Q}}(L) = -\frac{1}{L} \frac{4\pi^2 \sqrt{\lambda}}{\Gamma(1/4)^4} \quad (\text{for } T = 0). \quad (29)$$

Both  $\Delta S_{NG}$  and  $V_{Q\bar{Q}}$  match the results of Ref. [19] when expressed in terms of the appropriate 't Hooft coupling in the gauge theory.

## B. Heavy Quark Potential at finite temperature

We shall now proceed with the calculation of finite  $T$  corrections to the result above by expanding Eq. (20) in powers of  $\delta = y_*^4$ , assuming that  $\delta \ll (1/4 - \lambda_{GB})/|\lambda_{GB}|$ . This generalizes earlier results for  $\mathcal{N} = 4$  SYM [24, 25, 26] to non-zero  $\lambda_{GB}$ . The boundary condition (20) translates into

$$LT = \frac{1}{2\sqrt{\pi}} \delta^{1/4} a^2 \frac{\Gamma(3/4)}{\Gamma(5/4)} \left[ 1 - \frac{1}{5} \frac{\delta a^2}{\sqrt{1 - 4\lambda_{GB}}} \right]. \quad (30)$$

The limit  $\delta \rightarrow 0$  at fixed  $\lambda_{GB}$  provides the leading correction to the vacuum result from the previous section. Expressing  $\delta$  in terms of  $LT$ ,

$$\delta = \frac{16\pi^2}{a^8} (LT)^4 \left( \frac{\Gamma(5/4)}{\Gamma(3/4)} \right)^4 \left[ 1 + \frac{64\pi^2}{5a^6} \frac{(LT)^4}{\sqrt{1 - 4\lambda_{GB}}} \left( \frac{\Gamma(5/4)}{\Gamma(3/4)} \right)^4 \right] \quad (31)$$

leads to

$$U_h = U_* \frac{2\sqrt{\pi}}{a^2} LT \frac{\Gamma(5/4)}{\Gamma(3/4)} \left[ 1 + \frac{16\pi^2}{5a^6} \frac{(LT)^4}{\sqrt{1 - 4\lambda_{GB}}} \left( \frac{\Gamma(5/4)}{\Gamma(3/4)} \right)^4 \right]. \quad (32)$$

The regularized action for this configuration is given by

$$\Delta S_{NG} = -\frac{a U_*}{\pi \alpha' T} \frac{\sqrt{\pi} \Gamma(3/4)}{\Gamma(1/4)} \left[ 1 + \frac{\delta a^2}{2\sqrt{1 - 4\lambda_{GB}}} + \mathcal{O}(\delta^2) \right] \quad (33)$$

$$= -\frac{2\sqrt{\lambda}}{LT} \left( \frac{\Gamma(3/4)}{\Gamma(1/4)} \right)^2 \left[ 1 + \frac{24\pi^2}{5} \frac{(LT)^4}{a^6 \sqrt{1 - 4\lambda_{GB}}} \left( \frac{\Gamma(5/4)}{\Gamma(3/4)} \right)^4 \right] \quad (\text{for } LT \rightarrow 0). \quad (34)$$

In the last step we made use of Eqs. (31) and (32). At finite temperature we identify  $\Delta S_{NG}$  with the free energy of the  $Q\bar{Q}$  pair divided by the temperature<sup>2</sup>, and so Eq. (25) leads to

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<sup>2</sup> Note that the entropy  $S = -\partial F_{Q\bar{Q}}/\partial T$  for this configuration is indeed positive.

the following potential:

$$V_{Q\bar{Q}} = -\frac{2\sqrt{\lambda}}{L} \left( \frac{\Gamma(3/4)}{\Gamma(1/4)} \right)^2 \left[ 1 - \frac{72\pi^2}{5} \frac{(LT)^4}{a^6 \sqrt{1-4\lambda_{GB}}} \left( \frac{\Gamma(5/4)}{\Gamma(3/4)} \right)^4 \right] \quad (\text{for } LT \rightarrow 0) . \quad (35)$$

The first term coincides, of course, with the vacuum potential from Eq. (29) while the second term corresponds to the leading correction at small  $LT$ . Using Eq. (8), the potential can also be expressed in terms of  $\eta/s$

$$V_{Q\bar{Q}} = -\frac{2\sqrt{\lambda}}{L} \left( \frac{\Gamma(3/4)}{\Gamma(1/4)} \right)^2 \left[ 1 - \frac{576\pi^2}{5} \frac{(LT)^4}{\eta' (1+\eta')^3} \left( \frac{\Gamma(5/4)}{\Gamma(3/4)} \right)^4 \right] , \quad (36)$$

where  $\eta' \equiv \sqrt{4\pi \frac{\eta}{s}}$ . This expression applies when the second term in the square brackets is a small correction.

We observe that at fixed distance the potential decreases towards higher temperature (however, its gradient increases in magnitude). We compare to the behavior obtained from resummed pQCD where the  $Q\bar{Q}$  free energy at distances  $m_D L \ll 1$  is given by

$$F_{Q\bar{Q}} = -C_F \frac{g_{QCD}^2}{4\pi L} \left[ 1 - \left( 1 - \frac{\xi}{6} \right) m_D L + \frac{1}{2} \left( 1 - \frac{3\xi}{8} \right) m_D^2 L^2 + \dots \right] . \quad (37)$$

This expression follows from the Fourier transform of the resummed retarded propagator for static gluons [27, 28]. Here,  $m_D^2 = g^2 N_c T^2 / 3 = \lambda_{QCD} T^2 / 3$  denotes the square of the Debye screening mass at leading order. The parameter  $\xi$  is proportional to the product of  $\eta/s$ , expansion rate  $\Gamma$ , and inverse temperature and is assumed to be small <sup>3</sup> [29]:

$$\xi \sim \frac{\Gamma}{T} \frac{\eta}{s} . \quad (38)$$

If the entropy contribution is removed from eq. (37) then medium induced screening effects are pushed to order  $(m_D L)^2$  [28] and we obtain the following potential:

$$V_{Q\bar{Q}} = -C_F \frac{g_{QCD}^2}{4\pi L} \left[ 1 - \frac{1}{2} \left( 1 - \frac{3\xi}{8} \right) m_D^2 L^2 + \dots \right] . \quad (39)$$

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<sup>3</sup> For very heavy quarks the time scale associated with the heavy quark bound state,  $1/|E_{\text{bind}}|$ , is much shorter than the other time scales associated with temperature variations and the expansion rate. Thus, one can perform the calculations at fixed  $T$  and set  $\Gamma/T$  to be a constant. The AdS/CFT result in Eq. (36) should therefore be compared to the pQCD result assuming that  $\xi$  is on the order of  $\eta/s$  times a numerical coefficient.

In qualitative agreement with (36), the potential energy decreases (in magnitude) as  $T$ , and hence  $m_D$ , increases. There is also qualitative agreement between eqs. (36) and (39) in that the “screening corrections” (the second terms in the square brackets) decrease as  $\eta/s$  increases. However, note that the strong coupling result in Eq. (36) predicts a more rapid disappearance of temperature effects as  $LT \rightarrow 0$  than the perturbative QCD result shown in Eq. (39). The quartic dependence on  $LT$  in Eq. (36) (also found in Refs. [24, 25]) originates from the behavior of the metric near the horizon, i.e., the  $(U_h/U)^4$  term in Eq. (5)<sup>4</sup>.

The free energy of a single heavy quark  $F_Q$  in the plasma can also be obtained from the regularized action in Eq. (22). Due to conformal invariance, it should be expected that  $F_Q \sim T$  since  $T$  is the only energy scale available. In fact, one can simply take the limit  $U_* \rightarrow U_h$  in Eq. (22) (straight string limit) to show that

$$F_Q = -\frac{\sqrt{\lambda}}{1 + \eta'} T . \quad (40)$$

Hence,  $F_Q$  decreases in magnitude with increasing viscosity. This is qualitatively similar to the behavior obtained from resummed perturbation theory [28] where

$$F_Q = -\frac{1}{2}\alpha_s C_F m_D(T) \left(1 - \frac{\xi}{6} + \dots\right) \quad (41)$$

at small  $\xi$ . Note that both (40) and (41) are pure entropy contributions  $\sim TS_Q = -\partial F_Q / \partial \log T$  and so the potential energy of the quark in the plasma vanishes once that is removed, according to Eq. (25). The  $Q\bar{Q}$  potential at infinite separation,  $V_\infty$ , is therefore zero.

Lattice data for the free energy of a static  $Q\bar{Q}$  pair at infinite separation, in SU(3) Yang-Mills theory as well as for 2, 2+1 and 3-flavor QCD [30], can be parameterized as [23]

$$F_\infty(T) = 2F_Q(T) \simeq \frac{a}{T} - bT , \quad (42)$$

with  $a \approx 0.08 \text{ GeV}^2$  a constant of dimension two, not to be confused with  $a(\lambda_{GB})$  appearing in the metric (5), while  $b$  is a dimensionless number. The first term from Eq. (42) gives rise to a non-vanishing  $V_\infty(T)$  tied to the presence of an additional dimensionful scale besides  $T$ . In fact, for heavy quarks forming very small bound states, the temperature dependence of the

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<sup>4</sup> In general, for black Dp-branes in asymptotically AdS<sub>D</sub> spaces (note that  $D = p + 2 \geq 5$ ) the correction would be  $\sim (LT)^{p+1}$ .

*short-distance* potential is much smaller than that of  $V_\infty(T)$  [28]<sup>5</sup>. Hence, we are presently unable to determine the dominant temperature and viscosity dependent contribution to binding energies, which would require a theory with broken conformal invariance, perhaps along the lines of Ref. [31]<sup>6</sup>. Rather, in the following section we shall only compute the *eigenvalue* of the Hamiltonian.

It is also interesting to recall that the expectation value of a circular loop in  $\mathcal{N} = 4$  SYM at zero temperature is given by [34]

$$\langle W \rangle_{\text{circ}} = \exp \sqrt{\lambda}, \quad (43)$$

which agrees with our expression (40) if we identify the expectation value of the loop with  $\exp(-F_Q/T)$ , where  $1/T$  is the length of the loop in the Euclidean time direction; Eq. (40) also exhibits the dependence on the shear viscosity in the large- $N_c$  limit and at sufficiently large  $t'$  Hooft coupling  $\lambda$ .

#### IV. HEAVY QUARK BOUND STATES

At small  $t'$  Hooft coupling bound states of heavy quarks (“quarkonium”) have large Bohr radii  $a_0 \gg 1/M_Q$  as compared to the Compton wavelength of the quark and small binding energies  $|E_{\text{bind}}| \ll M_Q$  [35]. Therefore, a potential model applies and the energy levels of the states can be obtained from a Schrödinger equation. This is no longer the case if the  $t'$  Hooft coupling is very large. However, in practice one may take  $\lambda = g_{\text{YM}}^2 N_c \sim 5 - 10$  [18] and the numerical prefactor of the Coulomb-like  $\sim 1/L$  potential obtained via the AdS/CFT correspondence is smaller than unity. Applying a potential model may therefore provide qualitatively useful insight.

The heavy quark potential at  $L \rightarrow 0$  is purely  $\sim 1/L$  for both AdS/CFT and pQCD. As we saw in the previous section, the leading corrections to the heavy quark potential in

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<sup>5</sup> The binding energy of a quarkonium state is defined as the *eigenvalue* of the Hamiltonian relative to the potential at infinity (the latter corresponds to the sum of the potential energies of a  $Q$  and a  $\bar{Q}$  which do not interact with each other):  $E_{\text{bind}} = \langle \Psi | \hat{H} - V_\infty | \Psi \rangle - 2m_Q$ .

<sup>6</sup> In weakly coupled QCD, a contribution to the single-quark free energy of the form  $F_Q \sim a/T$  could be generated by adding a non-perturbative contribution  $m_G^2/(\vec{k}^2 + m_D^2)^2$  to the static gluon propagator;  $m_G^2$  is a constant of dimension two [32]. This also leads to a non-vanishing trace of the energy-momentum tensor [33].

AdS/CFT and pQCD have different powers of  $LT$ . We shall determine the energy levels for both cases and check how they depend on  $\eta/s$ .

At short distances  $V_{Q\bar{Q}}(r) = -A/r$ , where  $A = 4\pi^2\sqrt{\lambda}/\Gamma(1/4)^4$  in GB and  $A = \lambda_{QCD}/(8\pi)$  for pQCD at large  $N_c$ . The energy levels in the  $\sim 1/r$  potential are

$$E_n^{T=0} = -M_Q \frac{A^2}{4n^2} . \quad (44)$$

In what follows, we restrict ourselves to the  $n = 1$  ground state. The ‘‘Bohr radius’’ of quarkonium is  $a_0 = 2/(M_Q A) \ll 1/T$  at sufficiently large quark mass. The wave function is

$$\psi_0 = \frac{e^{-r/a_0}}{a_0^{3/2} \sqrt{\pi}} . \quad (45)$$

At finite temperature the potentials have the form

$$V_{Q\bar{Q}}(r) = -\frac{A}{r} [1 - B(rT)^\gamma] \quad (46)$$

where for AdS/CFT  $\gamma = 4$  and

$$B = \frac{72\pi^2}{5a^6\sqrt{1-4\lambda_{GB}}} \left( \frac{\Gamma(5/4)}{\Gamma(3/4)} \right)^4 \quad (47)$$

$$= \frac{576\pi^2}{5} \frac{1}{\eta'(1+\eta')^3} \left( \frac{\Gamma(5/4)}{\Gamma(3/4)} \right)^4 . \quad (48)$$

For pQCD  $\gamma = 2$  and

$$B = \frac{\lambda_{QCD}}{6} \left( 1 - \frac{3\xi}{8} \right) . \quad (49)$$

It is sufficient for our purposes here to compute the  $T$ -dependent shift of the energy to leading order

$$\Delta E = \frac{4ABT^\gamma}{a_0^3} \int_0^\infty dr r^{\gamma+1} e^{-2r/a_0} = AB \frac{(a_0 T)^\gamma}{2^\gamma a_0} \Gamma(2+\gamma) \quad (50)$$

and so the ground state energy level becomes

$$E = -M_Q \frac{A^2}{4} \left( 1 - \frac{4}{2^\gamma} \frac{B}{A} \frac{(a_0 T)^\gamma}{M_Q a_0} \Gamma(2+\gamma) \right) . \quad (51)$$

Substituting for  $A$ ,  $B$  and  $\gamma$  we obtain for GB

$$E_{GB} = E_{GB}^{T=0} \left[ 1 - \frac{C}{\lambda^2} \frac{1}{\eta'(1+\eta')^3} \frac{T^4}{M_Q^4} \right] , \quad (52)$$

where

$$C = \frac{27}{256} \frac{\Gamma^{24}(1/4)}{\pi^{10}} \approx 3 \times 10^7 , \quad (53)$$

is a numerical constant and where  $E^{T=0}$  denotes the ground state energy in the “Coulomb” potential given in Eq. (44).

On the other hand, in pQCD

$$E_{\text{pQCD}} = -M_Q \frac{\lambda_{QCD}^2}{256\pi^2} \left[ 1 - \frac{128\pi^2}{\lambda_{QCD}} \left( 1 - \frac{3\xi}{8} \right) \frac{T^2}{M_Q^2} \right] \quad (54)$$

$$= E_{\text{pQCD}}^{T=0} \left[ 1 - \frac{128\pi^2}{\lambda_{QCD}} \left( 1 - \frac{3\xi}{8} \right) \frac{T^2}{M_Q^2} \right]. \quad (55)$$

As expected, the  $T$ -dependent shifts in Eqs. (52,55) exhibit a different dependence on the  $t'$  Hooft coupling. However, the expression obtained from AdS/CFT also drops more rapidly with  $M_Q/T$  and with the viscosity than predicted by pQCD.

## V. SUMMARY AND OUTLOOK

We have determined the dependence of the static  $Q\bar{Q}$  potential on the temperature  $T$  and shear viscosity to entropy density ratio  $\eta/s$  in a conformal field theory dual to Gauss-Bonnet gravity on  $\text{AdS}_5$ . We found that, with increasing viscosity, the screening of the potential due to the thermal medium weakens and so the potential energy increases in magnitude. Moreover, the free energy of a single heavy quark decreases in magnitude with increasing viscosity. Both observations are in qualitative agreement with expectations from (“hard thermal loop” resummed) perturbative QCD.

In fact, at short distances the medium-induced effects on quarkonium binding energies are found to be very small, of order  $\sim (T/M_Q)^4 \times 1/\lambda^2 \eta'^4$ , where  $\eta' \equiv \sqrt{4\pi \eta/s}$ . The dominant temperature and viscosity dependence of the binding energies therefore arises due to the continuum threshold, i.e. from the value of the potential at  $L \rightarrow \infty$ . Both pQCD (at leading order) as well as exactly conformal gauge theories obtained using AdS/CFT, where  $T$  is the only dimensionfull scale available, can only generate a pure entropy contribution to the free energy of the  $Q\bar{Q}$  pair at infinite separation, and so  $V_\infty = 0$  in both cases. It would be interesting to construct a gravity dual for a field theory on the boundary with a contribution of the form  $\sim a/T$  to  $F_\infty$  as indicated by lattice QCD. This would provide a model for the dominant  $T$  and  $\eta$  dependence of quarkonium binding energies in a non-Abelian strongly-coupled plasma.

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